

Mass gap in quantum energy spectrum of relativistic Yang-Mills fields

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Abstract

A non-perturbative and mathematically rigorous quantum Yang-Mills theory on 4-dimensional Minkowski spacetime is set up in the framework of a complex nuclear Kree-Gelfand triple. It involves an infinite-dimensional symbolic calculus of operators with variational derivatives and a new kind of infinite-dimensional ellipticity.

In the temporal gauge and Schwinger first order formalism classical Yang-Mills equations become a semilinear hyperbolic system for which the general Cauchy problem (with no restriction at space infinity) is equivalent to one with a family of periodic initial data. Yang-Mills quartic self-interaction and the simplicity of a compact gauge Lie group imply that the energy spectrum of the anti-normal quantization of Yang-Mills energy functional of periodic initial data is a sequence of non-negative eigenvalues converging to infinity and, by caveat, has a mass gap at the spectral bottom. Furthermore, the energy spectrum (including the mass gap) is self-similar relative to an infrared cutoff: it is inversely proportional to the initial data period.

Key words: Second quantization; Kree-Gelfand nuclear triples; operators with variational derivatives; infinite-dimensional pseudodifferential calculus; Infinite-dimensional ellipticity; Yang-Mills Millennium problem.

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*In memoriam I. M. Gelfand (1913-2009)
on the occasion of his 100th anniversary.*

1 Introduction

This paper provides a solution for *both* parts of the 7th problem of Clay Mathematics Institute "Millennium Prize Problems":

Prove that for any compact simple global gauge group, a nontrivial quantum Yang-Mills theory exists on Minkowski time-space and has a positive mass gap (cp. JAFFE-WITTEN[1]).

(This formulation is from WITTEN[36, p. 23].)

A relativistic quantum field theory is concerned with the quantum energy spectrum. In particular, the gap at the spectral bottom defines the quantum mass of a field. Actually, in accordance with Yukawa principle, a positive mass gap is an experimental fact for weak and strong subnuclear forces: *A limited force range suggests its positive mass* (cp. Yukawa 1949 Nobel prize lecture). This is a *quantum* effect because, when the natural units of the light velocity c and Planck quantum action \hbar are set to 1, the energy component of the relativistic energy-momentum vector has the natural physical dimension of the reciprocal (Compton) length $[L^{-1}]$.

On a given energy scale three independent dimension units are chosen and all physical magnitudes become dimensionless.

The well known theorem that elliptic (pseudo)differential operators on compact manifolds have infinite discrete spectra (see, e.g. SHUBIN[31]) provides a mathematical hint for the quantum yang-Mills energy spectrum. However,

Mathematically, quantum field theory involves integration, and elliptic operators, on infinite-dimensional spaces. Naive attempts to formulate such notions in infinite dimensions lead to all sorts of trouble. To get somewhere, one needs the very delicate constructions considered in physics, constructions that at first sight look rather specialized to many mathematicians. For this reason, together with inherent analytical difficulties that the subject presents, rigorous understanding has tended to lag behind development of physics. (WITTEN[35, p.346])

This is in spite of the fact that quantum field theory was born immediately after quantum mechanics that already matured to a rigorous mathematical theory in J. von Neumann's monograph "Mathematische Grundlagen der Quantenmechanik" (1932).

Two kinds of quantum field theory are known:

- Classical theory of quantum fields, famously initiated by P. Dirac in 1927, deals with *operator-valued* solutions of *non-linear* hyperbolic equations with partial derivatives. Unfortunately, the non-bounded and non-commuting linear operator values seriously aggravate the mathematical meaning of non-linear terms of the equations. Besides, the non-linearity is ill-suited for the quantum superposition Dirac postulate.
- Quantum theory of classical fields deals with *functional* solutions of *linear* Schrödinger equations with variational derivatives on solution spaces of those hyperbolic equations. There was a vivid discussion among W. Heisenberg, P. Jordan, and W. Pauli of the corresponding "Volterra mathematics". E.g., P. Jordan and W. Pauli considered a one dimensional variational Schrödinger equation for eigenfunctionals $\Psi(\phi(x))$ of massless scalar fields $\phi(x), x \in \mathbb{R}$ (*Zur*

Quantumelectrodynamik ladungsfreier Felder, Zeitung für Physik, **47** (1928))

$$-\left(\frac{\hbar}{4\pi}\right)^2 \int dx \left[\frac{\delta^2}{\delta\phi(x)^2} + c^2 \left(\frac{d\phi(x)}{dx} \right)^2 \right] \Psi(\phi(x)) = \lambda \Psi(\phi(x)).$$

By *Bogoliubov-Shirkov-Schwinger quantization postulate* [7, chapter II], the variational Schrödinger operator is a quantization of the dynamical energy invariant, the integral of the initial data for Euler-Lagrange equation for classical fields.

In 1954 GELFAND-MINLOS[15] proposed to solve infinite-dimensional quantum fields equations with variational derivatives via approximations by solutions of partial differential equations with large but finite number of independent variables (cp. BEREZIN[4, Preface]). However the convergence of such approximations so far has not been established until the present paper.

The main content of the paper is split into three sections:

Section 2 provides a mathematically rigorous context for a non-perturbative quantum field theory on nuclear Kree-Gelfand triples. It includes the following items.

- Infinite-dimensional symbolic calculus of pseudovariational operators. Remarkably it involves entire sesqui-holomorphic expansions, not asymptotic ones as in finite dimensions.
- Convergence of approximations of pseudovariational operators by finite-dimensional pseudodifferential operators, along with the convergence of pseudodifferential symbolic calculi. This is a justification for Gelfand-Minlos solution [15] of equations with variational derivatives.
- New theory of infinite-dimensional elliptic operators including their spectral properties. The ellipticity means the operator domination of a power of the number operator. By Theorem 2.2 the spectrum of an elliptic non-negative operator is a sequence of non-negative eigenvalues converging to infinity. Thus it has positive a gap at the bottom.

Section 3 deals with the classical Yang-Mills theory:

- In the temporal gauge the Yang-Mills system of the second order partial differential equations for Yang-Mills connections on four dimensional Minkowski space is equivalent to the Schwinger semi-linear first order hyperbolic with constraint initial data on the Euclidean space \mathbb{R}^3 .
- By Ladyzhenskaya-Lax principle ([19] and [24]), the finite speed propagation of solutions of Yang-Mills system implies that the general Cauchy problem is

equivalent to ones with periodic Cauchy data. By GOGANOV-KAPITANSKII[19], Schwinger evolution system has unique smooth global solutions on Minkowski space with smooth constrained periodic initial data.

- Theorem 3.1 provides a rectification of the manifold of constrained periodic initial data allowing their parallel transport to Coulomb quasi gauge.

Section 4 presents the anti-normal quantization of the Yang-Mills energy functional in Gelfand-Kree triple over Coulomb quasi gauge. The main Theorem 3.1 affirms the ellipticity of the non-negative anti-normal quantum Yang-Mills operator and, therefore, discreteness of its spectrum. Proposition 4.1 exhibits the self-similarity of the energy spectrum relative to the running energy scale.

Section 4 also contains answers to physicists questions.

2 Elliptic pseudovariational operators

2.1 Review of Kree-Gelfand triples

Consider a Gelfand triple of densely imbedded complex topological spaces with the complex conjugation $*$ (see, e.g., GELFAND-VILENKIN[16])

$$\mathcal{H}^\infty \subset \mathcal{H}^0 \subset \mathcal{H}^{-\infty}, \quad (2.1)$$

where

- The space \mathcal{H}^0 is a Hilbert space with a Hermitian sesquilinear form z^*w :¹
- The space \mathcal{H}^∞ of elements z^* is a nuclear countably Hilbert space;
- The space $\mathcal{H}^{-\infty}$ of elements w is the anti-dual space of \mathcal{H}^∞ with respect to the Hermitian form z^*w .

Kree-Gelfand nuclear triple \mathcal{K} (KREE [20] and [21]) is a sesqui-holomorphic second quantization of the Gelfand triple \mathcal{H} , with the induced conjugation,

$$\mathcal{K}^\infty \subset \mathcal{K}^0 \subset \mathcal{K}^{-\infty} \quad (2.2)$$

where (cp., e.g., [12])

- The space $\mathcal{K}^{-\infty}$ is the countably Hilbert space of all entire holomorphic functionals $\Psi(z^*)$ on \mathcal{K}^∞ with the topology of compact convergence.²

¹The notation is bracketless as, e.g., in BEREZIN[4].

²A functional is entire on a locally convex complex vector space if it is continuous and entire on every complex line in that space (see, e.g., COLOMBEAU[10]).

- The space \mathcal{H}^0 is the Bargmann-Hilbert space of square integrable entire holomorphic functionals on $\mathcal{H}^{-\infty}$ with respect to the Gaussian probability measure (see, e.g., GELFAND-VILENKIN[15]). Bargmann Hermitian form

$$\langle \Psi | \Phi \rangle \equiv \int dz^* dz e^{-z^* z} \Psi^*(z) \Phi(z^*), \quad (2.3)$$

where the $*$ -dual $\Psi^*(z)$ is the complex conjugate of $\Psi(z^*)$.

- The space \mathcal{H}^∞ is the anti-dual of $\mathcal{H}^{-\infty}$.
- *Borel-Fourier transform* (see, e.g., COLOMBEAU[10, Chapter 7, Abstract])

$$\tilde{\Psi}(\zeta) \equiv \langle \Psi | e^\zeta \rangle, \quad \Psi \in \mathcal{H}^{-\infty}, \quad \tilde{\Psi}(z^*) \equiv \langle \Psi | e^{z^*} \rangle, \quad \Psi \in \mathcal{H}^\infty, \quad (2.4)$$

(where $e^\zeta(z^*) = e^{z^* \zeta} = e^{z^*}$) is a topological isomorphism between $\mathcal{H}^{-\infty}$ and the nuclear space of entire functionals $\Psi(\zeta)$ of exponential type on $\mathcal{H}^{-\infty}$.³

- The Borel-Fourier transform intertwines directional differentiation and multiplication

$$(\partial_w^* \Psi)^\sim = (w^* \zeta) \tilde{\Psi}, \quad (\partial_w \Psi^*)^\sim = (\zeta^* w) \tilde{\Psi}^*. \quad (2.5)$$

By Grothendieck kernel theory, the nuclearity of the Kree-Gelfand triples implies that the locally convex vector spaces $\mathbf{Q}(\dots \rightarrow \dots)$ of continuous linear operators are topologically isomorphic to the complete sesqui-linear tensor products (both spaces are endowed with the topology of compact uniform convergence).

$$\mathbf{Q}(\mathcal{H}^\infty \rightarrow \mathcal{H}^{-\infty}) \simeq \mathcal{H}^{*- \infty} \hat{\otimes} \mathcal{H}^{-\infty}, \quad (2.6)$$

$$\mathbf{Q}(\mathcal{H}^\infty \rightarrow \mathcal{H}^\infty) \simeq \mathcal{H}^{*- \infty} \hat{\otimes} \mathcal{H}^\infty, \quad (2.7)$$

$$\mathbf{Q}(\mathcal{H}^{-\infty} \rightarrow \mathcal{H}^\infty) \simeq \mathcal{H}^{* \infty} \hat{\otimes} \mathcal{H}^\infty, \quad (2.8)$$

where the operators are *tame* in the case of (2.7).

An operator is *polynomial* if its kernel is a continuous polynomial on $\mathcal{H}^{* \infty} \times \mathcal{H}^\infty$, so that polynomial operators are tame.

The formulas present the one-to-one correspondence between operators and the sesqui-holomorphic kernels of their matrix elements.

The nuclear Gelfand triple of the sesqui-Hermitian direct products

$$\mathcal{H}^{* \infty} \times \mathcal{H}^\infty \subset \mathcal{H}^{*0} \times \mathcal{H}^0 \subset \mathcal{H}^{*- \infty} \times \mathcal{H}^{-\infty} \quad (2.9)$$

carries the Hermitian conjugation

$$(z^*, w)^* \equiv (w^*, z) \quad (2.10)$$

³An entire functional is of exponential type if it has an exponential growth with respect to any continuous seminorm on $\mathcal{H}^{-\infty}$.

The associated Kree-Gelfand triples of sesqui-holomorphic kernels consists of

$$\mathcal{H}^{*\infty} \widehat{\otimes} \mathcal{H}^\infty \subset \mathcal{H}^{*0} \widehat{\otimes} \mathcal{H}^0 \subset \mathcal{H}^{*- \infty} \widehat{\otimes} \mathcal{H}^{-\infty} \quad (2.11)$$

where $\mathcal{H}^{*0} \widehat{\otimes} \mathcal{H}^0$ is the Hilbert space of Hilbert-Schmidt kernels.

The corresponding exponential functionals are

$$e^{(\zeta^*, \eta)}((z^*, w)^*) = e^{w^* \eta + \zeta^* z}. \quad (2.12)$$

2.2 Pseudovariational operators

Kree-Gelfand triple (2.2) has the canonical linear representation by continuous linear transformations of $\zeta \in \mathcal{H}^{-\infty}$ and $\zeta^* \in \mathcal{H}^\infty$ into the adjoint operators of *creation and annihilation* continuous operators of multiplication and directional differentiation

$$\hat{\zeta} : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, \quad \hat{\zeta} \Psi(z^*) \equiv (z^* \zeta) \Psi(z^*), \quad (2.13)$$

$$\widehat{\zeta^*} : \mathcal{H}^{-\infty} \rightarrow \mathcal{H}^{-\infty}, \quad \widehat{\zeta^*} \Psi(z) \equiv (\zeta^* z) \Psi(z), \quad (2.14)$$

$$\widehat{\zeta^*}^\dagger : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, \quad \widehat{\zeta^*}^\dagger \Psi(\zeta^*) \equiv \partial_{\zeta^*} \Psi(\zeta^*), \quad (2.15)$$

$$\hat{\zeta}^\dagger : \mathcal{H}^{-\infty} \rightarrow \mathcal{H}^{-\infty}, \quad \hat{\zeta}^\dagger \Psi(z) \equiv \partial_\zeta \Psi(z), \quad (2.16)$$

such that

1. Bosonic commutation relation

$$[\widehat{\zeta^*}^\dagger, \hat{\eta}] = \zeta^* \eta, \quad [\hat{\zeta}^\dagger, \widehat{\eta^*}] = \eta^* \zeta. \quad (2.17)$$

2. The exponentials e^η , $\eta^* \in \mathcal{H}^{-\infty}$, and e^η , $\eta \in \mathcal{H}^\infty$, are the eigenstates of the annihilation operators

$$\widehat{\zeta^*}^\dagger e^\eta = (\zeta^* \eta) e^\eta, \quad \hat{\zeta}^\dagger e^{\eta^*} = (\eta^* \zeta) e^{\eta^*}. \quad (2.18)$$

Creators and annihilators generate strongly continuous abelian operator groups of *quantum exponentials* in $\mathcal{H}^{-\infty}$ and \mathcal{H}^∞ parametrized by ζ and ζ^* :

$$e^{\hat{\zeta}} : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, \quad e^{\hat{\zeta}} \Psi(z^*) = e^{z^* \zeta} \Psi(z^*); \quad (2.19)$$

$$e^{\hat{\zeta}^\dagger} : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, \quad e^{\hat{\zeta}^\dagger} \Psi(z) = \Psi(z + \zeta); \quad (2.20)$$

$$e^{\widehat{\zeta^*}} : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, \quad e^{\widehat{\zeta^*}} \Psi(z) = e^{\zeta^* z} \Psi(z); \quad (2.21)$$

$$e^{\widehat{\zeta^*}^\dagger} : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, \quad e^{\widehat{\zeta^*}^\dagger} \Psi(z^*) = \Psi(z^* + \zeta^*). \quad (2.22)$$

Borel-Fourier transform of kernels $M(\theta^*, \eta)$

$$\tilde{M}(z^*, w) = \langle M(\theta^*, \eta) | e^{z^* \eta + \theta^* w} \rangle = \langle M(\theta^*, \eta) | \langle e^{z^*} | e^w \rangle \rangle \quad (2.23)$$

may be quantized as *normal*, *Weyl*, and *anti-normal pseudovariational operators* \widehat{M}

$$\widehat{M} : \mathcal{H}^{-\infty} \rightarrow \mathcal{H}^{-\infty}, \quad M \in \mathcal{H}^{*\infty} \hat{\otimes} \mathcal{H}^{\infty} \quad (2.24)$$

defined by their exponential matrix elements

$$\langle e^{z^*} | \widehat{M}_V | e^w \rangle \equiv \langle M_V(\theta^*, \eta) | \langle e^{z^*} | e^{\widehat{\theta}} e^{\widehat{\eta^*}^\dagger} | e^w \rangle \rangle, \quad (2.25)$$

$$\langle e^{z^*} | \widehat{M}_\omega | e^w \rangle \equiv \langle M_\omega(\theta^*, \eta) | \langle e^{z^*} | e^{\widehat{\theta} + \widehat{\eta^*}^\dagger} | e^w \rangle \rangle, \quad (2.26)$$

$$\langle e^{z^*} | \widehat{M}_\alpha | e^w \rangle \equiv \langle M_\alpha(\theta^*, \eta) | \langle e^{z^*} | e^{\widehat{\eta^*}^\dagger} e^{\widehat{\theta}} | e^w \rangle \rangle. \quad (2.27)$$

Proposition 2.1 *Any continuous linear operator Q from $\mathcal{H}^{-\infty}$ to $\mathcal{H}^{-\infty}$ has a unique normal co-kernel $M_V^Q(\zeta^*, \eta)$.*

PROOF Since

$$\langle e^{z^*} | e^{\widehat{\eta}} e^{\widehat{\theta^*}^\dagger} | e^w \rangle = \langle e^{\widehat{\eta}^\dagger} e^{z^*} | \widehat{\theta^*}^\dagger | e^w \rangle \quad (2.28)$$

$$= \langle e^{z^*} \eta e^{z^*} | e^{\theta^* z} e^w \rangle = e^{z^* \eta + \theta^* z} e^{z^* w}, \quad (2.29)$$

one has

$$\langle e^{z^*} | \widehat{M}_V | e^w \rangle = \langle M_V(\theta^*, \eta) | e^{z^* \eta + \theta^* z} e^{z^* w} \rangle = \widetilde{M}_V(z^*, w) e^{z^* w} \quad (2.30)$$

where $\widetilde{M}_V(z^*, w)$ is the sesqui-linear Fourier transform of $M_V(\theta^*, \eta)$ (see (2.12)). Thus $\widetilde{M}_V(z^*, w) e^{z^* w}$ is the kernel of \widehat{M}_V .

By (2.11), any Grothendieck kernel has a unique such representation. QED

By the Taylor expansion centered at the origin, the sesqui-entire functionals are uniquely defined by their restrictions to the real diagonal $(z^*, w = z) \in \Re(\mathcal{H}^{*\infty} \times \mathcal{H}^{\infty})$, so that the *normal symbol* of the operator Q

$$\sigma_V^Q(z^*, z) \equiv \widetilde{M}_V = \langle e^{z^*} | \widehat{M}_V e^z \rangle^Q(z^*, z) \quad (2.31)$$

exists and defines Q uniquely.

By Baker-Campbell-Hausdorff commutator formula and the canonical commutation relations (2.17),

$$e^{\widehat{\eta}} e^{\widehat{\theta^*}^\dagger} = e^{\widehat{\eta} + \widehat{\theta^*}^\dagger} e^{\theta^* \eta / 2}, \quad e^{\widehat{\theta^*}^\dagger} e^{\widehat{\eta}} = e^{\widehat{\eta} + \widehat{\theta^*}^\dagger} e^{-\theta^* \eta / 2}. \quad (2.32)$$

Thus any operator Q has Weyl and anti-normal co-kernels M_ω^Q and M_α^Q . Their restrictions to the real diagonal are *Weyl* and *anti-normal* symbols σ_ω^Q and σ_α^Q of Q .

By (2.32) the symbols of the same operator Q are related via *Weierstrass transform* (cp. AGARWAL-WOLF[2, formulas (5.29), (5.30), (5.31), page 2173] in a finite dimensional case; DYNIN [12]) in white noise calculus):

$$\sigma_\omega^Q(z^*, z) = e^{-(1/2)\partial_{z^*}\partial_z} \sigma_V^Q(z^*, z), \quad (2.33)$$

$$\sigma_\alpha^Q(z^*, z) = e^{-\partial_{z^*}\partial_z} \sigma_V^Q(z^*, z), \quad (2.34)$$

$$\sigma_\omega^Q(z^*, z) = e^{(1/2)\partial_{z^*}\partial_z} \sigma_\alpha^Q(z^*, z), \quad (2.35)$$

where the operator $e^{\pm(1/2)\partial_{z^*}\partial_z}$ is the sesqui-linear Fourier equivalent of the multiplication by $e^{\pm(1/2)z^*z}$, i. e.,

$$\partial_{z^*}\partial_z \equiv \text{Trace}(\partial_{z^*}\partial_w). \quad (2.36)$$

(Note that the multiplication by e^{z^*w} is a continuous operator in $\mathcal{H}^{-\infty*} \hat{\otimes} \mathcal{H}^{-\infty}$, so that the restrictions of $e^{\partial_{z^*}\partial_z} \sigma(\zeta^*, \zeta)$ to the real diagonal $\Re(\mathcal{H}^{\infty*} \times \mathcal{H}^{\infty})$ are well defined.)

The following Proposition shows that anti-normal operators are infinite-dimensional Berezin-Toeplitz integral operators from \mathcal{H}^{∞} to $\mathcal{H}^{-\infty*}$ (cp. BEREZIN[5, Equation (2.5)]).

Proposition 2.2 *The kernel of an anti-normal operator $Q = \hat{M}_\alpha$*

$$\langle e^{z^*} | Q | e^w \rangle = e^{z^*w} \sigma_\alpha^Q(w^*, w), \quad (2.37)$$

i.e. Q acts on $\Psi(w^)$ as the multiplication $\sigma_\alpha^Q(w^*, w)\Psi(w^*) \in \text{Ent}(\mathcal{H}^{\infty})^* \times \mathcal{H}^{\infty}$ followed by the orthogonal projection with the kernel $(e^{z^*w})^* = e^{w^*z}$ onto $\text{Ent}(\mathcal{H}^{\infty})$.*

PROOF Since

$$\langle e^{z^*} | e^{\widehat{\theta^*}} e^{\widehat{\eta}} | e^w \rangle = \langle e^{\widehat{\theta^*}} e^{z^*} | e^{\widehat{\eta}} e^w \rangle \quad (2.38)$$

$$= \langle e^{\theta^*z} e^{z^*} | e^{z^*\eta} e^w \rangle = e^{z^*w} e^{\theta^*w + w^*\eta}, \quad (2.39)$$

the kernel (2.27)

$$\langle M_\alpha(\theta^*, \eta) | e^{z^*w} e^{\theta^*w + w^*\eta} \rangle = e^{z^*w} \sigma_\alpha(w, w^*). \quad (2.40)$$

QED

Corollary 2.1 *The diagonal matrix elements of an anti-normal operator Q on \mathcal{H}^{∞}*

$$\langle e^{z^*} | Q | e^z \rangle \geq \inf \sigma_\alpha^Q(z^*, z), \quad (z^*, z) \in \mathcal{H}^{\infty}. \quad (2.41)$$

This Corollary is an infinite-dimensional extension of Theorem 7.1 in BEREZIN[5].

2.3 Quantized Galerkin approximations

A *Galerkin sequence* p_n , $j = 1, 2, \dots$, is an increasing sequence of tame orthogonal projections of rank n strongly convergent to the identity operator in \mathcal{H}^∞ . The projectors p_n are uniquely extended to the Galerkin families in \mathcal{H}^0 and $\mathcal{H}^{-\infty}$. The notation p_n is kept for all these extensions.

The finite dimensional projectors induce the *quantized Galerkin sequence*

$$P_n \Psi(z^*) \equiv \Psi(p_n z^*), \quad P_n \Psi(z) \equiv \Psi(p_n z) \quad (2.42)$$

of *infinite dimensional* projectors in the triple \mathcal{K} onto cylindrical triples isomorphic to the pulled back sesqui-entire triples over the tautological finite dimensional triple $\mathbb{C}^n \subset \mathbb{C}^n \subset \mathbb{C}^n$.

By Proposition 2.1, the compressions of operators $Q_n \equiv P_n Q P_n$ of Q are cylindrical pseudodifferential operators with the exponential kernels,

$$\langle e^{z^*} | Q_n | e^w \rangle = \langle e^{p_n z^*} | Q | e^{p_n w} \rangle, \quad (2.43)$$

i.e. pullbacks from \mathbb{C}^j of finite dimensional pseudodifferential operators of AGARWAL-WOLF[2].

Theorem 2.1 *Operator Q is the strong limit of the cylindrical pseudodifferential operators Q_n on \mathcal{K}^∞ .*

PROOF The matrix element $\langle \Psi^* | Q | \Phi \rangle$ is a separately continuous sesquilinear form on the Frechet space \mathcal{K}^∞ . By a Banach theorem (see, e.g., [27, v.1, Theorem V.7]), the sesquilinear form is actually continuous on $\mathcal{K}^{*\infty}$. In particular, operator Q is the weak limit of Q_n in $\mathcal{K}^{*\infty}$. Since $\mathcal{K}^{*\infty}$ is a nuclear space, the weak convergence implies the strong one in the topology of $\mathcal{K}^{*\infty}$. QED

As $n \rightarrow \infty$, the exponential matrix elements

$$\langle e^{z^*} | Q_n | e^w \rangle = \langle e^{p_n z^*} | Q | e^{p_n w} \rangle \longrightarrow \langle e^{z^*} | Q | e^w \rangle, \quad (2.44)$$

so that symbols of the cylindrical Q_n converge to the corresponding symbols of Q .

Thus if operators Q_1 and Q_2 are tame then (cp. AGARWAL-WOLF[2, Theorem III.5])

Corollary 2.2 *The symbols of the tame $Q_3 = Q_2 Q_1$ are convergent series (polynomials when Q_1 or Q_2 is a polynomial operator)*

$$\sigma_v^{Q_3}(z^*, z) = \sum_{m=0}^{\infty} (m!)^{-1} \partial_z^m \sigma_v^{Q_2}(z^*, z) \partial_{z^*}^m \sigma_v^{Q_1}(z^*, z), \quad (2.45)$$

$$\sigma_\alpha^{Q_3}(z^*, z) = \sum_{m=0}^{\infty} (m!)^{-1} \partial_{z^*}^m \sigma_\alpha^{Q_2}(z^*, z) \partial_z^m \sigma_\alpha^{Q_1}(z^*, z), \quad (2.46)$$

$$\sigma_\omega^{Q_3}(z^*, z) = \sum_{m=0}^{\infty} (m!)^{-1} \Omega^m (\sigma_\omega^{Q_2}(z_2^*, z_2) \sigma_\omega^{Q_1}(z_1^*, z_1)), \quad (2.47)$$

where $\Omega \equiv (1/2)(\partial_{z_2^*}\partial_{z_1} - \partial_{z_1^*}\partial_{z_2})$, $z_2^* = z_1^* = z^*$.

Another consequence from Theorem 2.1 for essentially selfadjoint

Corollary 2.3 *Let Q be the Friedrichs extension of a non-negative tame operator in $\mathcal{K}^{*\infty}$ to an (unbounded) selfadjoint operator in \mathcal{K}^0 . Then the spectrum of Q consists of limits of some spectral values of Q_n as n converges to infinity.*

PROOF By Theorem [27, v.1, Theorem VIII.25] and Theorem 2.1, the resolvent operators of Q_n strongly converge to the resolvent operator of Q .

Then [27, v.1, Theorem VIII.24] implies the lower semicontinuity of the spectra convergence. QED

2.4 Ellipticity

The tame quadratic *number operator* N is defined by its quadratic symbols

$$\sigma_v^N = z^*z + 1, \sigma_\alpha^N = z^*z, \sigma_\omega^N = z^*z + 1/2. \quad (2.48)$$

The notation N is transferred to the Friedrichs extension of the number operator. The eigenspaces \mathcal{N}_n , $n = 1, 2, \dots$, of N with the eigenvalues n are the spaces spanned by continuous homogeneous polynomials of degree n . The number operator is positive and self-adjoint in the Hilbert space \mathcal{K}^0 .

Thus N has the simple eigenvalue $\lambda_1 = 1$, and all other eigenvalues $\lambda_n = n$ form its essential spectrum.

A non-negative symmetric tame operator Q is *elliptic* ⁴ if there exist positive constants r and c such that

$$\langle \Psi | Q | \Psi \rangle \geq c \langle \Psi^* | N^r | \Psi \rangle, \quad \Psi \in \mathcal{K}^\infty. \quad (2.49)$$

The notation Q is transferred to the Friedrichs extension of the operator Q .

Theorem 2.2 *The spectrum of a positive elliptic operator Q is a sequence $\lambda_n(Q) \rightarrow +\infty$. In particular, Q has a mass gap at the bottom of its spectrum.*

PROOF Quantized Galerkin cutoffs $P_j Q P_j$ of elliptic non-negative operators Q in \mathcal{K}^∞ are pullbacks of the finite dimensional elliptic pseudodifferential operators Q_j on $p_j X$. The latter are non-negative and elliptic, so that they have discrete spectra of eigenvalues $\lambda_n(Q_j) \geq 0$ converging to infinity (see, e.g. SHUBIN[31, Theorem 26.3]). Then the operators $P_j Q P_j$ have the infinitely degenerated spectra of Q_j .

Similarly, for $k \geq j$, the operators $p_j Q_k p_j$ on $p_k X$ are pullbacks of Q_j , so that Q_j have trivial extensions \check{Q}_j of Q_j (with $\check{Q}_j = 0$ on the orthogonal complement $(p_j X)^\perp$

⁴The definitions of ellipticity in [3], [23], and [12] are not sufficient for the present paper.

in $p_k X$. Again, both \check{Q}_j and Q_j have the same spectra (though with different but finite multiplicities).

Since $Q_k \geq \check{Q}_j$, a variational principle (see, e. g. BEREZIN-SHUBIN[6, Appendix 1, Corollary 2 of Proposition 3.1])⁵ implies that the eigenvalues $\lambda_n(Q_j)$ monotonically increase with j . Then, by Corollary 2.3, the spectrum of the elliptic operator Q consists of some of the limits of these monotonic sequences.

The cutoffs $P_j N P_j$ of the number operator are pullbacks of the finite dimensional pseudodifferential number operators $N_j = p_j N p_j$ on $p_j X$. All they have the same spectra \mathbb{Z}_+ , so that $p_j c N^r p_j$ with positive c and r have discrete spectra cn^r . Now the same variational principle and ellipticity condition (2.49) imply that the eigenvalues $\lambda_n(Q_j) \geq cn^r$. Therefore, for any given integer n there may be only finitely many limits of monotonic $\lambda_n(Q_j)$ as j increases to infinity. QED

3 Classical Yang-Mills theory

3.1 Yang-Mills fields

The *global gauge group* \mathbb{G} of a Yang-Mills theory is a connected semi-simple compact Lie group with the Lie algebra $\text{Ad}(\mathbb{G})$.

The notation $\text{Ad}(\mathbb{G})$ indicates that the Lie algebra carries the adjoint representation $\text{Ad}(g)X = gXg^{-1}$, $g \in \mathbb{G}$, $a \in \text{Ad}(\mathbb{G})$, of the group \mathbb{G} and the corresponding self-representation $\text{ad}(X)Y = [X, Y]$, $X, Y \in \text{Ad}(\mathbb{G})$. Then $\text{Ad}(\mathbb{G})$ is identified with a Lie algebra of skew-symmetric matrices and the matrix commutator as Lie bracket with the *positive definite* Ad-invariant scalar product

$$X \cdot Y \equiv \text{Trace}(X^T Y), \quad (3.1)$$

where $X^T = -X$ denotes the matrix transposition.

Let the Minkowski space \mathbb{M} be oriented and time oriented with the Minkowski metric signature $(-1, 1, 1, 1)$. In a Minkowski coordinate systems x^μ , $\mu = 0, 1, 2, 3$, the metric tensor is diagonal. In the natural unit system, the time coordinate $x^0 = t$. Thus $(x^\mu) = (t, x^i)$, $i = 1, 2, 3$.

The *local gauge group* \mathcal{G} is the group of infinitely differentiable \mathbb{G} -valued functions $g(x)$ on \mathbb{M} with the pointwise group multiplication. The *local gauge Lie algebra* $\text{Ad}(\mathcal{G})$ consists of infinitely differentiable $\text{Ad}(\mathbb{G})$ -valued functions on \mathbb{M} with the pointwise Lie bracket.

\mathcal{G} acts via the pointwise adjoint action on $\text{Ad}(\mathcal{G})$ and correspondingly on \mathcal{A} , the real vector space of *gauge fields* $A = A_\mu(x) \in \text{Ad}(\mathcal{G})$.

⁵ If non-negative operators $A' \leq A''$ are essentially selfadjoint on a common domain D with non-decreasing sequential discrete spectra λ'_n and λ''_n , then $\lambda'_n \leq \lambda''_n$.

Gauge fields A define the *covariant partial derivatives*

$$\partial_{A\mu}X \equiv \partial_\mu X - \text{ad}(A_\mu)X, \quad X \in \text{Ad}(\mathcal{G}). \quad (3.2)$$

This definition shows that in the natural units *gauge connections have the mass dimension* $1/[L]$.

Any $g \in \mathcal{G}$ defines the *affine gauge transformation*

$$A_\mu \mapsto A_\mu^g := \text{Ad}(g)A_\mu - (\partial_\mu g)g^{-1}, \quad A \in \mathcal{A}, \quad (3.3)$$

so that $A^{g_1}A^{g_2} = A^{g_1g_2}$.

Yang-Mills *curvature tensor* $F(A)$ is the antisymmetric tensor ⁶

$$F(A)_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \quad (3.4)$$

The curvature is gauge invariant:

$$\text{Ad}(g)F(A) = F(A^g), \quad (3.5)$$

as well as *Yang-Mills Lagrangian*

$$(1/8)F(A)^{\mu\nu} \cdot F(A)_{\mu\nu}. \quad (3.6)$$

The corresponding gauge invariant Euler-Lagrange equation is a 2nd order non-linear partial differential equation $\partial_{A\mu}F(A)^{\mu\nu} = 0$, called the *Yang-Mills equation*

$$\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = 0. \quad (3.7)$$

Yang-Mills fields are solutions of Yang-Mills equation.

In the temporal gauge $A_0(t, x^k) = 0$ ⁷ the 2nd order Yang-Mills equation (3.7) is equivalent to the 1st order Schwinger hyperbolic system for the time-dependent $A_j(t, x^k), E_j(t, x^k)$ on \mathbb{T}^3 (see, e.g., GOGANOV-KAPITANSKII [19, Equation (1.3)])

$$\partial_t A_k = E_k, \quad \partial_t E_k = \partial_j F_k^j - [A_j, F_k^j], \quad F_k^j = \partial^j A_k - \partial_k A^j - [A^j, A_k]. \quad (3.8)$$

and the *constraint equations*

$$[A^k, E_k] = \partial^k E_k, \quad \text{i.e.} \quad \partial_{k,A} E_k = 0. \quad (3.9)$$

⁶The dimensionless Yang-Mills coupling g_{YM}^2 is set to 1

⁷I. Segal theory ([30]) of infinite-dimensional Sobolev Lie groups implies that for any infinitely differentiable gauge field on \mathbb{M} with periodical Cauchy data there is a unique infinitely differentiable gauge transformation to the temporal gauge.

3.2 Schwinger first order formalism

Rewrite the 2nd order Yang-Mills equations (3.7) in the temporal gauge $A_0(t, x^k) = 0$ as the 1st order systems of the *evolution equations* for the time-dependent $A_j(t, x^k)$, $E_j(t, x^k)$ on \mathbb{R}^3 as

$$\partial_t A_k = E_k, \quad \partial_t E_k = \partial_j F_k^j - [A_j, F_k^j], \quad F_k^j = \partial^j A_k - \partial_k A^j - [A^j, A_k]. \quad (3.10)$$

and the *constraint equations*

$$[A^k, E_k] = \partial^k E_k, \quad \text{i.e.,} \quad \partial_{k,A} E_k = 0 \quad (3.11)$$

By GOGANOV-KAPITANSKII [19], the evolution system is a semilinear first order partial differential system with *finite speed propagation* of the initial data, and the Cauchy problem for it with constrained initial data at $t = 0$

$$a_k(x) \equiv A(0, x_k), \quad e_k(x) \equiv E(0, x_k), \quad \partial^k e_k = [a_k, e_{,k}] \quad (3.12)$$

is *globally and uniquely solvable* in local Sobolev spaces on the whole Minkowski space \mathbb{M} (with no restrictions at the space infinity.)

This fundamental theorem has been derived via Ladyzhenskaya method (1949, not published, cp. LAX[24] for linear hyperbolic systems) by a reduction to the case of *T-periodic* Cauchy data on 3-dimensional tori $\mathbb{T}^3 \in \{x_k \leq \pi T\}$.

If the constraint equations are satisfied at $t = 0$, then, in view of the evolution system, they are satisfied for all t automatically. Thus the *1st order evolution system along with the constraint equations for Cauchy data is equivalent to the 2nd order Yang-Mills system*. Moreover the constraint equations are invariant under *time independent* gauge transformations. As the bottom line, we have

Proposition 3.1 *In the temporal gauge Yang-Mills fields A are in one-one correspondence with their gauge transversal Cauchy data (a, e) satisfying the equation $\partial_a e = 0$.*

3.3 Coulomb quasi-gauge of constrained Cauchy data

Consider the chain of Sobolev-Hilbert spaces \mathcal{A}^s , $-\infty < s < \infty$, of (generalized) connections $a(x)$ on a torus \mathbb{T}^3 of period T with the Hilbert norms $|a|_s$:

$$|a|_s^2 \equiv (1/T) \int_{\mathbb{T}^3} d^3x (a \cdot (1 - \Delta)^s a) < \infty. \quad (3.13)$$

They define the real Gelfand nuclear triple (cp., e.g., [15])

$$\mathcal{A} : \mathcal{A}^\infty \equiv \bigcap \mathcal{A}^s \subset \mathcal{A}^0 \subset \mathcal{A}^{-\infty} \equiv \bigcup \mathcal{A}^s, \quad (3.14)$$

where \mathcal{A}^∞ is a nuclear countably Hilbert space with the dual $\mathcal{A}^{-\infty}$.

Similarly we define the chain of Sobolev-Hilbert spaces $\mathcal{S}^s, -\infty < s < \infty$, of (generalized) Lorentz scalar fields $u(x)$ on \mathbb{T}^3 with values in $\text{Ad } \mathbb{G}$ and the Hilbert norms $|u|_s$. Let

$$\mathcal{S} : \mathcal{S}^\infty \equiv \bigcap \mathcal{S}^s \subset \mathcal{S}^0 \subset \mathcal{S}^{-\infty} \equiv \bigcup \mathcal{S}^s \quad (3.15)$$

be the corresponding Gelfand triple.

Let $a \in \mathcal{A}^{s+3}, s \geq 0$. Then, by Sobolev embedding theorem a is continuously $s+2$ -differentiable on \mathbb{T}^3 and, therefore, the following gauged vector calculus operators are continuous:

- *Gauged gradient*

$$\text{grad}_k^a u \equiv \partial_k u - [a_k, u] : \mathcal{S}^{s+1} \rightarrow \mathcal{A}^s, \quad (3.16)$$

- *Gauged divergence*

$$\text{div}^a b \equiv \partial_k b_k - [a_k, b_k] : \mathcal{A}^{s+1} \rightarrow \mathcal{S}^s, \quad (3.17)$$

- *Gauged Laplacian*

$$\Delta^a u \equiv \text{div}^a(\text{grad}^a u) : \mathcal{S}^{s+2} \rightarrow \mathcal{S}^s. \quad (3.18)$$

The gauge operators grad^a and div^a are sqew-adjoint with respect to the real $*$ -scalar product (3.13) with $s = 0$

$$(\text{grad}^a)^* u = -u^*(\text{div}^a b). \quad (3.19)$$

Lemma 3.1 *If $a \in \mathcal{A}^{s+3}, s \geq 0$, then the range of the operator div^a is the orthogonal complement of the finite-dimensional space of constant gauge fields (the same for all a and s).*

PROOF The gauged Laplacian $\text{div}_a \text{grad}_a$ differs from the usual Laplacian Δ^0 by first order differential operators. Therefore the operator product Δ^a and a right inverse of Δ^0 is a Fredholm integral operator in \mathcal{S}^s and therefore, has the closed range of a finite co-dimension. Since the range of div^a contains the range of the product, it is closed as well. By the positive definiteness of the Ad-invariant scalar product on the simple Lie algebra $\text{Ad } \mathbb{G}$ and by (3.19), the co-dimension of the range of div^a is equal to the nullity of the operators grad^a .

If $\Delta^a u = 0$ then the identity $(\Delta^a u)^* u = (\text{grad}^a u)^*(\text{grad}^a u)$ implies $\text{grad} u = [a, u]$ on the null space, so that

$$(1/2)(\text{grad} u)^* u = [a, u]^* u = \int_{\mathbb{T}} d^3 x [-\text{Trace}(auu - uau)] = 0. \quad (3.20)$$

Thus solutions $u \in \mathcal{S}^s$ are constant. QED

Similarly to (3.14), define the real Gelfand nuclear triple \mathcal{E} of gauge electric fields e with the Sobolev norms

$$|a|_s^2 \equiv T \int_{\mathbb{T}^3} d^3x (a \cdot (1 - \Delta)^s a) < \infty. \quad (3.21)$$

Consider the bundles $\mathcal{C}^s, s \geq 0$ of constraint Cauchy data with the base \mathcal{A}^∞ and the null spaces \mathcal{E}_a^{s+1} of the operators $\text{div}^a : \mathcal{E}^{s+1} \rightarrow \mathcal{E}^s$ as fibers over $a \in \mathcal{A}^\infty$.

Their intersection \mathcal{C}^∞ is a bundle of nuclear countably Hilbert spaces over the nuclear countably Hilbert base \mathcal{A}^∞ . Together with the unions of the dual spaces \mathcal{C}^{-s} they form a bundle of nuclear Gelfand triples \mathcal{C} over the same base.

Theorem 3.1 *The bundle \mathcal{C}^∞ is smoothly trivial, so that the total space of \mathcal{C}^∞ is smoothly isomorphic to the direct product of its base \mathcal{A}^∞ and the fiber $\mathcal{C}_{a=0}^\infty$, the nullspace of the operator div in \mathcal{E}^∞ .*

PROOF For $0 \leq s \leq \infty$ consider the mapping

$$f : \mathcal{A}^{s+2} \times \mathcal{E}^{s+1} \rightarrow \mathcal{A}^s, \quad f(a, e) \equiv \text{div}_a(e) \quad (3.22)$$

Sobolev imbedding theorem shows that the mapping is continuous. Lemma 3.1 implies that the continuous partial Frechet derivatives $\partial_e f(a, e)$ are bounded linear operators onto a fixed Hilbert space $T(s)$, the orthogonal complement of constant a 's. continuously dependent on the parameter $a \in \mathcal{A}^{s+2}$. The restrictions of $\partial_e f(a, e)$ to the orthogonal complements of the null spaces of div_a are one-to-one. By the implicit function theorem on Hilbert spaces (see, e.g., [22]), this implies that the explicit solutions $e = e(a)$ of the equation $f(a, e) = 0$ provide infinitely smooth local trivializations of Hilbert bundles \mathcal{C}^s .

Their intersection $\mathcal{C}^\infty = \cap \mathcal{C}^s$ is a locally trivial C^∞ -bundle over \mathcal{A}^∞ with the associated locally trivial bundle of smooth $*$ -orthonormal frames in the fibers.

Since \mathcal{A}^∞ is a Frechet space, its smooth homothety retraction to the origin $a = 0$ has a homotopy lifting to the frame space. Thus the bundle \mathcal{C}^∞ is trivial⁸, so that the total set of constraint Cauchy data carries the global chart $\mathcal{A}^\infty \times \mathcal{C}_{a=0}^\infty$. QED

Let \mathcal{A}^s and \mathcal{E}^s denote the nullspaces of the operator div in \mathcal{A}^s and \mathcal{E}^s .

By DELL'ANTONIO-ZWANZIGER[11], the closures of smooth gauge orbits in $\mathcal{H}^0 \equiv \mathcal{A}^0$ intersect \mathcal{A}^0 . These closures are the orbits of the Sobolev group, the closure in Sobolev space $W^{1,2}(\mathbb{T}^3)$ of the group of smooth gauge transformations. (The Sobolev group is a topological group of continuous transformations in \mathcal{A}^0 .) Thus $\mathcal{H}^0 \equiv \mathcal{A}^0 \times \mathcal{E}^0$ is a *quasi-gauge* for the orbifold of the direct product of the parallel transports (i.e. every $(a, e) \in \mathcal{H}^0$ is on an orbit but some orbits may intersect \mathcal{H}^0 more than once (cp. SINGER[32] and NARASIMHAN-RAMADAS[26])). The Gelfand

⁸Cp. BOSS-BLEECKER[9, page 67]

triple

$$\mathcal{H} : \mathcal{H}^\infty \equiv \mathcal{A}^\infty \times \mathcal{E}^\infty \subset \mathcal{H}^0 \equiv \mathcal{A}^0 \times \mathcal{E}^0 \subset \mathcal{H}^{-\infty} \equiv \mathcal{A}^{-\infty} \times \mathcal{E}^{-\infty} \quad (3.23)$$

is the direct product of the Gelfand triples \mathcal{A} and \mathcal{E} .

4 Anti-normal quantum Yang-Mills theory

4.1 Quantum Yang-Mills energy spectrum

The non-negative Noether energy functional of smooth Yang-Mills Cauchy data (cp. GLASSEY-STRAUSS[17, Section 3]) on \mathbb{T}^3 is

$$M(a, e) \equiv (1/2) \int_{\mathbb{T}^3} d^3x \left((\text{curl } a - [a, a]) \cdot (\text{curl } a - [a, a]) + e \cdot e \right). \quad (4.1)$$

Remark 4.1 *The density $\text{curl } a - [a, a] \cdot (\text{curl } a - [a, a])$ is the curvature of smooth gauge fields a , and, as such, is invariant under smooth gauge parallel transport but the density $e \cdot e$ is not.*

At the same time the density $e \cdot e$ is invariant under the flat isometric parallel transport provided by Theorem 3.1. Thus the energy functional M is constant on smooth orbits of the direct product of both parallel transports.

For smooth a

$$(\text{curl } a - [a, a]) \cdot (\text{curl } a - [a, a]) = \text{curl } a \cdot \text{curl } a + [a, a] \cdot [a, a] + 2a \cdot [\text{div } a, a]. \quad (4.2)$$

Therefore for $(a, e) \in \mathcal{H}^\infty$ (so that $\text{div } a = 0$) we have

$$M(a, e) = (1/2) \int_{\mathbb{T}^3} d^3x (\text{curl } a \cdot \text{curl } a + [a, a] \cdot [a, a] + e \cdot e). \quad (4.3)$$

Convert the Coulomb quasi-gauge triple 3.23 into the complex Gelfand triple \mathcal{H} with conjugation where the real and imaginary parts are the direct factors

$$\Re \mathcal{H} \equiv \mathcal{A}, \quad \Im \mathcal{H} \equiv \mathcal{E}. \quad (4.4)$$

Let the polynomial energy functional $M(z, z^*) \equiv M(a, e)$ (ref: eq:Noether) be the anti-normal symbol of the tame quantum Yang-Mills energy operator $\hat{M}_\alpha : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$.

Theorem 4.1 *The (unique) non-negative selfadjoint Friedrichs extension of \hat{M}_α in \mathcal{M}^0 is elliptic. Its spectrum is an infinite sequence of non-negative eigenvalues. In particular, the spectrum has a positive mass gap.*

PROOF (A) Let b_i be a basis for $\text{Ad}(\mathbb{G})$ with $b_i \cdot b_j = \delta_{ij}$. Then, by the *simplicity of the global gauge group*, the structure constants $c_{ij}^k = [b_i, b_j] \cdot b_k$ are anti-symmetric under interchanges of i, j, k . Thus if $a = a^i b_i \in \text{Ad}(\mathbb{G})$ then

$$a \cdot a = \text{Trace}(a^t a) = -a^i c_{ij}^k a^l c_{kj}^i = a^i c_{ij}^k a^l c_{lj}^k, \quad (4.5)$$

so that

$$[a, a] \cdot [a, a] = a^i a^j a^l a^m [b_i, b_j] \cdot [b_l, b_m] = a^i a^j a^l a^m c_{ij}^k c_{lm}^k = \sum_k (a^i a^j c_{ij}^k)^2. \quad (4.6)$$

Since $z = (1/\sqrt{2})(a + ie)$, the operator $\partial_z^* \partial_z = \partial_a^2 + \partial_e^2$, so that

$$e^{\partial_z^* \partial_z / 2}([a, a] \cdot [a, a]) \stackrel{(4.5)}{=} [a, a] \cdot [a, a] / 2 + a \cdot a. \quad (4.7)$$

Then the Weyl symbol of the operator \widehat{M}_α

$$\sigma_\omega^H(a, e) \stackrel{(2.35), (4.7)}{=} \int_{\mathbb{T}^3} d^3x ([a, a] \cdot [a, a] / 2 + a \cdot a + e \cdot e) + 1/2. \quad (4.8)$$

(B) The Weyl quantization of $[a, a] \cdot [a, a]$ is the operator of multiplication with $[a, a] \cdot [a, a] \geq 0$ in the $”(a, e)$ -representation” of the canonical commutation relations (cp. AGARWAL-WOLF [2, Section VII, page 2177]). In particular, the Weyl multiplication operator is non-negative.

(C) By (2.48),

$$(1/2) \int_{\mathbb{T}^3} d^3x (a \cdot a + e \cdot e) + 1/2$$

is the anti-normal symbol of the number operator N .

Altogether we get the operator inequality of non-negative operators

$$\widehat{M}_\alpha \geq \widehat{N}_\alpha. \quad (4.9)$$

Now the spectral Theorem 2.2 implies the Theorem 4.1. QED

Proposition 4.1 *The spectra of Yang-Mills quantum energy operators are self-similar: they are inversely proportional to the initial data periods.*

PROOF The scaling transformation

$$\check{x} \equiv x/T, \quad \check{a} \equiv a/T, \quad \check{e} \equiv e/T^2, \quad (4.10)$$

converts the energy functional (4.1) over \mathbb{T}^3

$$(1/2) \int_{\mathbb{T}^3} d^3x ((\text{curl } a - [a, a]) \cdot (\text{curl } a - [a, a]) + e \cdot e) \quad (4.11)$$

with a period T into the scaled energy functional over $\check{\mathbb{T}}^3$

$$(1/2T) \int_{\check{\mathbb{T}}^3} d^3x ((\text{curl } a - [a, a]) \cdot (\text{curl } a - [a, a]) + e \cdot e) \quad (4.12)$$

with the period 1, and the scalar product with the period T

$$\int_{\mathbb{T}_1^3} d^3x (a \cdot a/T + Te \cdot e) \quad (4.13)$$

into the scalar product over $\check{\mathbb{T}}^3$ with the period 1.

QED

4.2 FAQ

- **Q:** How the constrained Yang-Mills energy functional differs from the singular Yang-Mills hamiltonian?

A: The proposed quantum Yang-Mills theory is of Lagrangian, not of Hamiltonian variety. It deals with quantization of Noether energy functional via Schwinger-Bogoliubov-Shirkov quantum action principle on the rectified manifold of constrained Cauchy data, not with the a quantization of the constraint Hamiltonian dynamics via Feynman diagrams⁹

- **Q:** Is the derived quantum Yang-Mills energy spectrum relativistically invariant?

A: No. Yang-Mills energy functional is only a time component of the relativistic time-like Noether energy-momentum. (The classical relativistic Yang-Mills mass is the positive square root of negative Lorentz square of the latter.)

- **Q:** Is the derived quantum Yang-Mills energy spectrum gauge invariant?

A: No. By Remark 4.1, the energy functional is not gauge invariant.

- **Q:** Where are divergences, renormalization, BRST, and other trappings of perturbative quantum field theory?

A: The proposed mathematically rigorous quantum Yang-Mills theory is decidedly non-perturbative.

- **Q:** Relativistic Yang-Mills action integral is conformally invariant on the Minkowski space, and therefore cannot be compatible with a positive mass gap. How the conformal invariance of the action functional is broken?

A: The energy functional is an integral over \mathbb{T}^3 , and therefore has the reciprocal length dimension (in the natural physical units $c = 1$, $\hbar = 1$).

⁹ Classically, by MONCRIEF[25] there is an equivalence between constrained Yang-Mills energy functional and singular Yang-Mills hamiltonian (cp.FADDEEV-SLAVNOV [13, Section 2, Chapter III]).

- **Q:** The official formulation of the Millennium Yang-Mills problem [1] is looking for a quantum Yang-Mills theory with axiomatic properties at least as strong as axioms of Wightman relativistic quantum field theory. Why there is no mention of these axioms in the paper?

A: Even modified Wightman axioms (see, e.g., BOGOLIUBOV et al [7, chapter 10] are in a serious conflict with the simplest cases of Gupta-Bleuler theory of quantum electromagnetic fields, as well as commonly used local renormalizable gauges (see, e.g. STROCCI[33, Chapter 6 and Appendix A.2]). Occam's razor is appropriate.

- **Q:** How it is possible to solve a millennium problem in few pages?

A: "Grothendieck is famous for his mystical conviction that a mathematical problem will solve itself when, by a sufficient humble attentiveness, one has found exactly its right context and formulation" ([28]). Incidentally, nuclear spaces, discovered by Grothendieck in 1950's, immediately led to numerous applications in the framework of nuclear Gelfand triples. Certainly the proposed solution of the Millennium 7th problem has been reached from shoulders of giants.

- **Q:** How this mathematical theory is related to conventional perturbative theory?

A: By Proposition 4.1, the spectrum and, in particular, the mass gap depend on the running coupling Tg_{YM}^2 , as prescribed by the renormalization group in the perturbative quantum Yang-Mills theory, e.g. in the ultraviolet by dimensional transmutation (cp. FADDEEV[14]).

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